UNIT 11: THE RIEMANN INTEGRAL

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11.1 LEARNING OBJECTIVES

In this unit, we will be able to learn about :

- partitions and Riemann sums
- upper and lower Riemann integrals
- riemann integrability
- riemann necessary and sufficient condition for R-integrability
- some classes of integrable functions
- fundamental theorem of integral calculus.

11.2 INTRODUCTION

The integration theory have arisen due to the necessity to evaluate the areas of plane regions. Likewise, integration of any function is interpreted to be the area under the graph of the function. The idea was to divide the plane region into various rectangles and find the limit of the sum of those rectangles when number of rectangles tend to infinity. This reference to integration from summation view point was always associated with

geometrical concepts. To formulate an independent theory of integration, the German mathematician, Riemann, gave a purely arithmetic treatment to the subject and thus developed the subject entirely free from the intuitive dependence on geometrical concepts. Afterwards, many developments and generalizations of the theory of integration followed. In this chapter, we define integration based on the definition by Riemann. However, this chapter is limited to finding integration over intervals and not any other sets. The function we discuss in this chapter is assumed to be bounded and unbounded functions are beyond the scope of this chapter. Accordingly, we begin by discussing the integration of real bounded functions on intervals and conclude the chapter by establishing the Fundamental theorem of Integral Calculus.

11.3 RIEMANN INTEGRABILITY

We begin by defining partition of an integral $I := [a,b]$.

Definition: Let $I := [a,b]$ is a closed bounded interval in R. Then a partition of [*a, b*] is a finite, ordered set $P := (x_0, x_1, ..., x_n - 1, x_n)$ of points in [*a, b*] such that

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

The points of *P* divide [*a, b*] into non-overlapping sub-intervals

$$I_1 := [x_0, x_1], I_2 := [x_1, x_2], ..., I_n := [x_{n-1}, x_n]$$

We use the symbol Δx_i to denote the sub-length $x_i - x_{i-1}$ of the i^{th} sub-interval $[x_{i-1}, x_i]$, i.e.

$$\Delta x_i = x_i - x_{i-1}, (i = 0,1,..., n)$$

As mentioned in the Introduction, in this chapter, we take function *f* to be a bounded real-valued function on [*a, b*]. Trivially, *f* is bounded on each sub-interval corresponding to each partition *P*. Let M_i and m_i be the supremum and infimum of *f* in $[x_{i-1}, x_i]$.

We define Upper and Lower sum of *f* corresponding to partition *P* as follows:

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 + ... + M_n \Delta x_n$$

and

$$L(P,f)=\sum_{i=1}^{n} m_i\Delta x_i = m_1\Delta x_1 + m_2\Delta x_2 + ... + m_n\Delta x_n$$

If *M* and *m* are the supremum and infimum of *f* in [*a, b*], then

$$m \le m_i \le M_i \le M$$

which implies

$$m\Delta x_i \le m_i\Delta \le M_i\Delta \le M\Delta$$

Putting *i* = 1, 2, ..., *n* and adding all the inequalities, we get

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

(1)

For each partition *P*, there exist *L* (*P, f*) and *U* (*P, f*). We denote *U* to be the set of all upper sums of *f* for all the partitions of [*a, b*] i.e.

$$U = \{U(P,f) : P \text{ is a partition of } [a, b]\}$$

We denote *L* to be the set of all lower sums of *f* for all the partitions of [*a, b*] i.e.

$$L = \{L(P,f) : P \text{ is a partition of } [a, b]\}$$

Since both U(P, f) and L (P, f) are bounded as seen from (1), so the set *U* and *L* have supremum and infimum. The infimum of the set of upper sums is called the Upper integral and the supremum of the set of lower sums is called the Lower integral over [*a, b*] i.e.

$$\int_a^{\bar{b}} f dx = \inf U$$

and

$$\int_{\bar{a}}^{b} f dx = \sup L$$

Definition: The function *f* on [*a, b*] is said to be Riemann Integrable over [*a, b*] if

$$\int_a^{\bar{b}} f dx = \int_{\bar{a}}^{b} f dx = c\,(\text{say})$$

and the common value *c* of the integral is called the Riemann integral of *f* over [*a, b*] denoted as $\int_a^b f dx$.

Note: If *f* is integrable over [*a, b*], then it can be expressed as $f \in R[a,b]$.

Remark: The upper and lower integrals are defined for every bounded function but they may not necessarily be equal for every bounded function.

Functions whose upper and lower integrals are not equal are said to be non-integrable functions.

Example: Show that a constant function *f*(*x*) = *k* is integrable over [*a, b*] i.e. $f \in R[a,b]$.

For any partition *P* of the interval [*a, b*], we have

$$\begin{aligned} L(P,f) &= k\Delta x_1 + k\Delta x_2 + ... + k\Delta x_n \\ &= k\left(\Delta x_1 + \Delta x_2 + ... + \Delta x_n\right) \\ &= k(b-a) \end{aligned}$$

and $$\begin{aligned} U(P,f) &= k\Delta x_1 + k\Delta x_2 + ... + k\Delta x_n \\ &= k(b-a) \end{aligned}$$

So $$\int_{\underline{a}}^{b} k dx = \sup L(P,f) = k(b-a)$$

and $$\int_{a}^{\overline{b}} k dx = \inf U(P,f) = k(b-a)$$

Thus $\int_{\underline{a}}^{b} k dx = \int_{a}^{\overline{b}} k dx = k(b-a)$

which indicates that the function *f* (*x*) = *k* is integrable and

$$\int_{a}^{b} k dx = k(b-a)$$

Definition: For any partition *P*, the length of the largest sub-interval is called the norm of the partition and is denoted by $\mu(P)$, i.e.

$$\mu(P) = \max \Delta x_i \ (1 \le i \le n)$$

Definition: A partition *P** is said to be refinement of *P* if $P \subseteq P^*$ i.e. every point of P is a point of *P**.

Theorem : If *P** is a refinement of a partition *P*, then for a bounded function *f*,

(i) $L(P^*, f) \ge L(P,f)$

(ii) $U(P^*, f) \le U(P,f)$

Proof: (i) As the function is bounded over the entire interval [*a, b*], it is bounded in every sub-interval $[x_{i-1},\ x_i],\ (i = 1,2,...,n)$.

Let us suppose *P** contains just one point more than *P* and let this extra point be ξ where $x_{i-1} < \xi < x_i$ for some *i*.

Let $\omega_1,\ \omega_2,$ and m_i be the infimum of *f* in the intervals $[x_{i-1}, \xi], [\xi, x_i], [x_{i-1}, x_i]$ respectively.

Clearly, $m_i \le \omega_1$ and $m_i \le \omega_2$.

Thus,

$L(P^*, f) - L(P, f) = \omega_1(\xi - x_{i-1}) + \omega_2(x_i - \xi) - m_i(x_i - x_{i-1})$

$= (\omega_1 - m_i)(\xi - x_{i-1}) + (\omega_2 - m_i)(x_i - \xi) \geq 0$ since

$\xi - x_{i-1} \geq 0,\ x_i - \xi \geq 0$

Thus $L(P^*, f) \leq L(P, f)$

If P^* contains p points more than P, same steps can be repeated p times to arrive at

$$L(P^*, f) \leq L(P, f)$$

Similarly we can prove that $U(P^*, f) \geq U(P, f)$

Corollary: For a refinement P^* of P containing more than p points more than P and $|f(x)| \leq k$ for all $x \in |a, b|$,

$$L(P, f) \leq L(P^*, f) \leq L(P, f) + 2pk\mu(P)$$

$$U(P, f) \geq U(P^*, f) \geq U(P, f) - 2pk\mu(P)$$

Proof: Proceeding as in the above theorem, when P^* contains one point more than P, we get

$$L(P^*, f) - L(P, f) = (\omega_1 - m_i)(\xi - x_{i-1}) + (\omega_2 - m_i)(x_i - \xi)$$

Since $|f(x)| \leq k,$ for all $x \in [a, b]$, therefore

$$-k \leq m_i \leq \omega_1 \leq k$$

$$\Rightarrow 0 \leq \omega_1 - m_i \leq 2k$$

Similarly,

$$0 \leq \omega_2 - m_i \leq 2k$$

$$\Rightarrow L(P^*, f) - L(P, f) \leq 2k(\xi - x_{i-1}) + 2k(x_i - \xi)$$

$$= 2k\Delta x_i$$

$$\leq 2k\mu(P)$$

Now by introducing additional points one after another, and repeating the above reasoning p times, we get,

$$L(P^*, f) \leq L(P, f) + 2pk\mu$$

and since $L(P, f) \leq L(P^*, f)$

$$\Rightarrow \quad L(P, f) \leq L(P^*, f) \leq L(P, f) + 2pk\mu$$

Similarly the other result may be proved.

Theorem: For oany two partitions P_1 and P_2,

$$L(P_1, f) \leq U(P_2, f)$$

i.e. no upper sum can be less than any lower sum.

Proof: Let P^* be a common refinement of P_1, P_2 so that

$$P^* = P_1 \cup P_2$$

Using the above theorem, we get

$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_2, f)$$

Thus $L(P^*, f) \leq L(P, f)$

Corollary: For any bounded function f,

$$\underline{\int} f dx \leq \overline{\int} f dx$$

Proof: Keeping P_2 fixed and taking supremum over all partitions P_1 of the above theorem, we get,

$$\underline{\int} f dx \leq U(P_2, f)$$

Similarly, taking infimum over all P_2 in the above equation, we get

$$\underline{\int} f dx \leq \overline{\int} f dx$$

Theorem: Darboux's Theorem: If f is a bounded function on [a, b], then for every $\in > 0$, there corresponds $\delta > 0$ such that

(i) $U(P, f) < \overline{\int_a^b} f dx + \in$

(ii) $L(P, f) > \underline{\int_a^b} f dx - \in$

for every partition P of [a, b] with norm $\mu(P) < \delta$.

Proof: (i) Since f is bounded, therefore, there exists $k > 0$ such that

$$|f(x)| \leq k, \forall x \in [a, b]$$

Again, since the upper integral is the infimum of the set of upper sums, therefore for any $\in > 0$, there exists a partition $P_1 = \{x_0, x_1, ..., x_p\}$ of [a, b] such that

$$U(P_1, f) < \overline{\int_a^b} f dx + \frac{1}{2} \in \qquad (2)$$

The partition P_1 has $p - 1$ points besides $x_0 (= a)$ and $x_p (= b)$.

Let δ be a positive number such that $2k(p-1)\delta = \frac{1}{2} \in$.

Let P be any partition with norm $\mu(P) < \delta$ and P^* be a refinement of P and P_1 such that $P^* = P \cup P_1$.

Since P^* is a refinement of P having at most $p - 1$ more points than P, and $|f(x)| \leq k$ for all $x \in [a, b]$ so we get from above results,

$$U(P,f)-2k(p-1)\mu(P)\le U\ (P^*,\ f)$$

Also since $U(P,f)-2k(p-1)\delta \le U(P,f)-2k(p-1)\mu(P),$ we get, for any partition P with norm $\mu(P)<\delta,$

$$U(P,f)-2k(p-1)\delta \le (P^*,\ f)$$

$$\le U(P_1,f)$$

$$< \int_a^{\bar{b}} f dx+\frac{1}{2}\in \text{ (using equation (2))}$$

$$\Rightarrow U(P,f)<\int_a^{\bar{b}} f dx+\frac{1}{2}\in -\frac{1}{2}\in =\int_a^b f dx+\in$$

since $2k(p-1)\delta=\frac{1}{2}\in$

The other part can be proved similarly.



CHECK YOUR PROGRESS 1

Q 1: Show that $U(P^*, f) - L(P^*, f) \le U(P, f) - L(P, f)$ where P^* is a refinement of the partition P.

11.4 RIEMANN SUMS

In the previous section, we learned about integral of a function in terms of upper and lower sums. The numbers M_i and m_i appearing in these sums are not necessarily the values of the function f. However, in this section, we show that $\int f dx$ can also be considered as the limit of a sequence of sums in which M_i and m_i are replaced by the values of f.

Definition: Let P be a partition of [a, b]. Let us choose points $t_1, t_2, ..., t_n$ such that $x_{i-1} \le t_i \le x_i,\ (i = 1,2,...,n)$. The sum

$$S(P,F)=\sum_{i=1}^{n} f(t_i)\Delta x_i$$

is said to be Riemann sum of f over [a, b] relative to P.

Here t_i are arbitrary and it can be any point whatsoever of Δx_i.

Definition: If for every $\in > 0,$ there exist $\delta > 0$ such that

$$|S(P,f)-A|<\in$$

for every partition $P=\{x_0,x_1,...,x_n\}$ of [a, b] with norm $\mu(P)<\delta$ and for every choise of points $t_i \in [x_{i-1}, x_i],$ then $S(P, f)$ is said to converge to A as $\mu(P)\to 0$ i.e.

$$\lim_{\mu(P)\to 0} S(P,f) = A.$$

Definition: The second definition of integrability A function f is said to be integrable on [*a, b*] if $\lim_{\mu(P)\to 0} S(P,f)$ exists as $\mu(P)\to 0$ and

$$\lim_{\mu(P)\to 0} S(P,f) = \int_a^b f dx$$

Note: Since $\mu(P)\to 0$ when $n\to\infty,$ therefore $\lim_{\mu(P)\to 0}$ can be replaced by $\lim_{n\to\infty}$ in the above definition.

The equivalence of the two definitions of integrability will now be established as follows :

Let us suppose, a bounded function *f* is integrable according to the first definition of integrability i.e.

$$\int_a^{\bar{b}} f dx = \int_{\bar{a}}^b f dx = \int_a^b f dx$$

We need to show that the function *f* is integrable according to the second definition of integrability i.e. for any $\in>0,$ there exist $\delta>0$ such that for every partition *P* with $\mu(P)<\delta,$

$$|S(P,f)-A|<\in$$

$$\Rightarrow \left|\sum_{i=1}^n f(t_i)\Delta x_i - \int_a^b f dx\right| <\in$$

Now, let $\in>0$ be any positive number

By Darboux's theorem, there exists $\delta>0$ such that

$$U(P,f) < \int_a^{\bar{b}} f dx = \int_a^b f dx + \in$$

(3)

$$L(P,f) > \int_{\bar{a}}^b f dx - \in = \int_a^b f dx - \in \qquad (4)$$

for every partition *P* with $\mu(P)<\delta.$

If t_i is any point of $\delta x_i,$ we have

$$L(P,f) \le \sum_{i=1}^n f(t_i)\Delta x_i \le U(P,f) \qquad (5)$$

From (3), (4) and (5), we obtain that for any $\in>0,$ there exist $\delta>0$ such that for every partition *P* with $\mu(P)<\delta,$

$$\int_a^b f dx - \in < \sum_{i=1}^n f(t_i)\Delta x_i < \int_a^b f dx + \in$$

$$\Rightarrow \left| \sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f dx \right| < \in$$

Thus the function *f* is integrable according to the second definition of integrability.

Now, let us suppose the function *f* which is integrable according to the second definition of integrability i.e.

$\lim_{\mu(P)\to 0} S(P,f)$ exists

Thus for every $\in > 0,$ there exists $\delta > 0$, such that for every partition $P = \{x_0, x_1, ..., x_n\}$ with norm $\mu(P) < \delta$ and for every choice of point t_i in Δx_i, there exist a number *A*, such that

$$\left| \sum_{i=1}^{n} f(t_i)\Delta x_i - A \right| < \in$$

It will be first shown that *f* is bounded.

Let if possible, *f* be not bounded.

Taking $\in = 1,$ there exists a partition *P* such that for every choice of t_i in Δx_i,

$$\left| \sum_{i=1}^{n} f(t_i)\Delta x_i - A \right| < 1$$

$$\Rightarrow \left| \sum_{i=1}^{n} f(t_i)\Delta x_i - A \right| < |A| + 1$$

As *f* is not bounded in [*a,b*], it must also be so in at least one sub-interval say Δx_m.

Let us take $t_i = x_i$ when $i \neq m$ so that every t_i except t_m is fixed and accordingly every term of the sum $\sum f(t_i)\Delta x_i$ except the term $f(t_m)\Delta x_m$ is also fixed. Since *f* is not bounded in Δx_m, we can choose a point t_m in Δx_m such that

$$\left| \sum f(t_i)\Delta x_i \right| > |A| + 1$$

and thus we arrive at a contradiction.

Hence the function *f* is bounded on [*a,b*].

Now, let $\in$ be any positive number. So there exist $\delta > 0$ such that for all partition *P* with $\mu(P) < \delta,$ we have

$$A-\frac{1}{2}\in < S(P,f) < A+\frac{1}{2}\in \qquad (6)$$

We choose one such *P*. Taking the point t_i range over the intervals Δx_i and taking the supremum and infimum of the numbers *S*(*P, f*) obtained as such, we get from (6),

$$A-\frac{1}{2}\in < L(P,f) \le I(P,f) < A+\frac{1}{2}$$

$$\Rightarrow U(P,f)-L(P,f) < \in$$

Also since

$$L(P,f) \le \int_{\bar{a}}^{b} f dx \le \int_{a}^{\bar{b}} f dx \le U(P,f)$$

Therefore

$$\int_{a}^{\bar{b}} f dx - \int_{\bar{a}}^{b} f dx \le U(P,f) - L(P,F) < \in \qquad (7)$$

But $\in$ is an arbitrary positive number.

Thus (7) implies,

$$\int_{a}^{\bar{b}} f dx - \int_{\bar{a}}^{b} f dx = 0$$

$$\Rightarrow \int_{a}^{\bar{b}} f dx = \int_{\bar{a}}^{b} f dx$$

so that the function is integrable.

Example : Compute $\int_{a}^{b} f dx$ where $f(x) = 2x+1.$

Let $P = \{1 = x_0, x_1, ..., x_n = 2\}$ be a partition which divides [1, 2] into *n* equal sub-intervals,, each of length $\frac{2-1}{n} = \frac{1}{n},$ so that

$$\mu P = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

$$x_i = 1 + \frac{i}{n},\ i = 1,2,...,n$$

$$\Delta x_i = \frac{1}{n},\ i = 1,2,...,n$$

$$\sum_{i=1}^{n} \Delta x_i = n.\frac{1}{n} = 1$$

Let $t_i = x_i$ when $i = 1,2,...,n,$ then

$$S(P,f) = \sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} (2x_i + 1)\Delta x_i$$

$$= \sum_{i=1}^{n} \left\{ 2\left(1+\frac{i}{n}\right)+1 \right\} \Delta x_i = 3\sum_{i=1}^{n} \Delta x_i + \frac{2}{n^2}\sum_{i=1}^{n} i$$

$$= 3 + \frac{2n(n+1)}{n^2\ 2} = 3 + \frac{2}{2n}$$

Thus when $\mu(P) \to 0,$

$$\lim_{\mu(P)\to 0} S(P,f) = 3$$

Since the limit exists, so the function is integrable and

$$\int_a^b f dx = \lim_{\mu(P)\to 0} S(P,f) = 3$$



CHECK YOUR PROGRESS 2

Q 1: Show that $\int_a^b f dx = 1$ where $f(x) = |x|$.

Q 2: Show that $\int_a^b f dx = 11/2$ where $f(x) = 3x + 1$.

11.5 RIEMANN'S CONDITION FOR INTEGRABILITY

In this section, we discuss the necessary and sufficient conditions for the integrability of a bounded function *f* is that to everey $\in > 0,$ there corresponds $\delta > 0$ such that for every partition *P* of [*a, b*] with norm $\mu(P) < \delta,$

$$U(P,f) - L(P,f) < \in$$

Proof: Necessary part: Let us suppose that the bounded function *f* is integrable i.e.

$$\int_{\bar{a}}^{b} f dx = \int_a^{\bar{b}} f dx = \int_a^b f dx$$

Let $\in > 0$ be any number. By Darboux's Theorem, there exists $\delta > 0$ such that for every partition *P* with norm $\mu(P) < \delta,$ we have

$$U(P,f) < \int_a^{\bar{b}} f dx + \frac{1}{2} \in = \int_a^b f dx + \frac{1}{2} \in$$

(8)

$$L(P,f) > \int_{\bar{a}}^{b} f dx - \frac{1}{2} \in = \int_a^b f dx - \frac{1}{2} \in$$

(9)

Or

$$-L(P,f) < -\int_{\bar{a}}^{b} f dx + \frac{1}{2} \in$$

(10)

Adding equation (8) and (10), we get

$$U(P,f)-L(P,f)<\in$$

for every partition *P* with norm $\mu(P)<\delta$.

Sufficient part : Let $\in>0$. For any partition *P* with norm $\mu(P)<\delta$, we have

$$U(P,f)-L(P,f)<\in$$

For any partition, *P*, we know that

$$L(P,f)\le\int_{\bar{a}}^{b} f dx\le\int_{a}^{\bar{b}} f dx\le U(P,f)$$

$$\Rightarrow\int_{a}^{\bar{b}} f dx-\int_{\bar{a}}^{b} f dx\le U(P,f)-L(P,f)<\in$$

But $\in$ is an arbitrary positive number which can be as small as possible but greater than 0, so

$$\int_{a}^{\bar{b}} f dx-\int_{\bar{a}}^{b} f dx=0$$

$$\Rightarrow\int_{a}^{\bar{b}} f dx=\int_{\bar{a}}^{b} f dx$$

Thus *f* is integrable.

Note: Alternatively we can state that : A necessary and sufficient condition for the integrability of a bounded function *f* is that

$$\lim\{U(P,f)-L(P,f)\}=0$$

when the form $\mu(P)$ of the partition *P* tend to 0.

Theorem 2: A bounded function *f* is integrable on | *a, b*| if and only if for every $\in>0$, there exists a partition *P* of | *a, b*| such that

$$U(P,f)-L(P,f)<\in$$

Proof: Necessary part : Suppose the function *f* is integrable i.e.

$$\int_{a}^{\bar{b}} f dx=\int_{\bar{a}}^{b} f dx=\int_{a}^{b} f dx$$

Let $\in>0$. Since the upper and lower integrals are the infimum and supremum respectively of the upper and lower sums, therefore $\exists$ partitions P_1 and P_2 such that

$$U(P_1,f)<\int_{a}^{\bar{b}} f dx+\frac{1}{2}\in=\int_{a}^{b} f dx+\frac{1}{2}\in$$

and

$$L(P_2,f)>\int_{a}^{b} f dx-\frac{1}{2}\in=\int_{a}^{b} f dx-\frac{1}{2}\in$$

Let *P* be the common refinement of P_1 and P_2 i.e. $P=P_1\cup P_2$

Thus

$$U(P,f) \le U(P_1,f) < \int_a^b f dx + \frac{1}{2} \in < L(P_2,f) + \in \le L(P,f) + \in$$

Hence there exists a partition *P* such that $U(P,f) - L(P,f) < \in$

Sufficient part: Let $\in > 0$.

Let *P* be the partition for whch

$$U(P,f) - L(P,f) < \in$$

For any partition *P*, we have

$$L(P,f) \le \int_{\underline{a}}^{b} f dx \le \int_a^{\bar{b}} f dx \le U(P,f)$$

Thus

$$\int_a^{\bar{b}} f dx - \int_{\underline{a}}^{b} f dx \le U(P,f) - L(P,f) < \in$$

But $\in$ is an arbitrary number which can be as small as possible but greater than 0. Henceforth

$$\int_a^{\bar{b}} f dx = \int_{\underline{a}}^{b} f dx$$

which implies *f* is integrable.

11.6 SOME CLASSES OF INTEGRABLE FUNCTIONS

In this section we establish the Riemann integrability of some important classes of functions.

Theorem: Every continuous function on [*a, b*] is integrable on [*a, b*].

Proof: For $\in > 0,$ let us choose $\eta > 0$ such that

$$\eta(b-a) < \in$$

Since *f* is continuous on [*a, b*], therefore it is bounded and is uniformly continuous on [*a, b*] i.e. there exists $\delta > 0$ such that

$$|f(t_1) - f(t_2)| < \eta \text{ if } |t_1 - t_2| < \delta \text{ and } t_1, t_2 \in [a,b]$$

(11)

We now choose a partition *P* with norm $\mu(P) < \delta.$ Then by equation (11), we have $M_i - m_i \le \eta\,(i = 1,2,...,n).$ Thus

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i$$

$$\le \eta \sum_{i=1}^{n} \Delta x_i$$

$$= \eta(b-a) < \in$$

Henceforth *f* is integrable by Theorem 2.

Theorem: If a function *f* is monotonic on [*a, b*], then it is integrable on [*a, b*].

Proof : A monotonic function can either be increasing or decreasing. Let us suppose the function *f* is increasing.

For $\in > 0,$ let us choose η such that $\eta < \frac{\in}{f(b) - f(a) + 1}$

We choose a partition $P = \{x_0, x_1, ..., x_n\}$ of [*a, b*] with norm $\mu(P) < \eta$.

Since *f* is monotonic increasing, thus $M_i = f(x_i),\ m_i = f(x_{i-1}), (i = 1, 2, ..., n)$

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i$$

$$= \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\}\Delta x_i$$

$$< \eta \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\}\Delta x_i$$

$$< \frac{\in}{f(b) - f(a) + 1} \{f(b) - f(a)\}$$

$$< \in$$

Hence *f* is integrable.

11.7 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

We now prove a theorem known as the first Fundamental theorem of Calculus.

Theorem: If function *f* is bounded and integrable on [*a, b*], then the function *F* defined as

$$F(x) = \int_a^x f(t)dt, a \le x \le b$$

is continuous on [*a, b*], and furthermore, if *f* is continuous at a point $c \in [a, b]$, then *F* is derivable at *c* and

$$F'(c) = f(c)$$

Proof: Since function *f* is bounded, therefore there exists a number *M* > 0 such that

$$| f(x) | \le M,\ \forall x \in [a, b]$$

If x_1 and x_2 are two points of [*a, b*] such that $a \le x_1 < x_2 \le b,$ then

$$|F(x_2) - f(x_1)| = \left| \int_{x_1}^{x_2} f(t)dt \right|$$

$$\le M(x_2 - x_1)$$

Thus for any $\in > 0,$ we have

$$F(x_2) - F(x_1)| < \in, \text{ if } |x_2 - x_1| < \in / M$$

Hence the function *F* is continuous on [*a, b*].

Now, given *f* is continuous at a point $c \in [a,b]$ i.e. for any $\in > 0,$ there exists $\delta > 0,$ such that

$$|f(x) - f(c)| < \in, \text{whenever } |x - c| < \delta$$

Let $c - \delta < s \le c \le t < c + \delta$

Thus

$$\left| \frac{F(t) - f(s)}{t - s} - f(c) \right| = \left| \frac{1}{t - s} \int_s^t \{f(x) - f(c)\} dx \right|$$

$$\le \frac{1}{t - s} \int_s^t |f(x) - f(c)| dx < \in$$

Thus $F'(c) = f(c)$ i.e. continuity of *f* at any point of [*a, b*] implies derivability of *F* at that point.

Note: Since *c* is any point of [*a, b*], we have for all $x \in [a,b]$,

$$F'(t) = f(t) \Rightarrow F' = f$$

i.e. continuity of *f* on [*a, b*] implies derivability of *F* on [*a, b*]

Definition: A derivable function *F*, if it exists such that its derivative is F' is equal to a given function *f* is called a primitive of *f*.

Remark: Continuity of a function is not a necessary condition for the existence of a primitive.

Another theorem we state below is said to be the Second Fundamental theorem of Calculus.

Theorem: If a function *f* is bounded and integrable on [*a, b*] and there exists a function *F* such that $F' = f$ on [*a, b*], then,

$$\int_a^b f dx = F(b) - F(a)$$

Proof: The function $F' = f$ is bounded and integrable, i.e. for any given $\in > 0,$ there exists a $\delta > 0$ such that for every partition $P = \{a = x_0, x_1, ..., x_n = b\}$ with norm $\mu(P) < \delta,$

$$\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f dx\right| < \in$$

Or

$$\lim_{\mu(P)\to 0}\sum_{i=1}^{n} f(t_i)\Delta x_i = \int_a^b f dx \qquad (12)$$

for every choice of points t_i in Δx_i.

We choose t_i in Δx_i in such a way that by Lagrange's Mean Value theorem, we have

$$\begin{aligned} F(x_i) - f(x_{i-1}) &= F'(t_i)\Delta x_i,\ (i = 1,2,...,n) \\ &= f(t_i)\Delta x_i \\ \Rightarrow \sum_{i=1}^{n} f(t_i)\Delta x_i &= \sum_{i=1}^{n}\{F(x_i) - F(x_{i-1})\} \\ &= F(b) - F(a) \end{aligned}$$

Hence from (12), we get

$$\int_a^b f dx = F(b) - F(a)$$



11.8 LET US SUM UP

- Let $I := [a,b]$ is a closed bounded interval in R. Then a partition of [*a, b*] is a finite, ordered set $P := (x_0, x_1, ..., x_{n-1}, x_n)$ of points in [*a, b*] such that

  $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$

  The points of *P* divide [*a, b*] into non-overlapping sub-intervals

  $I_1 := [x_0, x_1],\ I_2 := [x_1, x_2],\ ...,\ I_n := [x_{n-1}, x_n]$

  We use the symbol Δx_i to denote the sub-length $x_i - x_{i-1}$ the i^{th} sub-interval $[x_{i-1}, x_i]$, i.e.

  $\Delta x_i = x_i - x_i,\ (i = 0,1,...,n)$

- Let M_i and m_i be the supremum and infimum of *f* in $[x_{i-1}, x_i]$.

  We define Upper and Lower sum of *f* corresponding to partition *P* as follows:

  $$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = M_1\Delta x_1 + M_2\Delta x_2 + ... + M_n\Delta x_n$$

and

$$L(P,f)=\sum_{i=1}^{n} m_i\Delta x_i = m_1\Delta x_1 + m_2\Delta x_2 + ... + m_n\Delta x_n$$

- The infimum of the set of upper sums is called the Upper integral and the supremum of the set of lower sums is called the Lower integral over [*a, b*] i.e.

$$\int_a^{\bar{b}} f dx = \inf U$$

and $$\int_{\bar{a}}^{b} f dx = \sup L$$

- The function *f* on [*a, b*] is said to be Riemann Integrable over [*a, b*] if

$$\int_a^{\bar{b}} f dx = \int_{\bar{a}}^{b} f dx = c \text{ (say)}$$

and the common value *c* of the integral is called the Riemann integral of *f* over [*a, b*] denoted as $\int_a^b f dx.$

- For any partition *P*, the length of the largest sub-interval is called the norm of the partition and is denoted by $\mu(P),$ i.e.

$$\mu(P) = \max \Delta x_i \, (1 \le i \le n)$$

- A partition *P** is said to be refinement of P if $P \subseteq P^*$ i.e. every point of *P* is a point of *P**.
- *f P** is a refinement of a partition *P*, then for a bounded function *f*,

(i) $L(P^*, f) \ge L(P,f)$

(ii) $U(P^*, f) \le U(P,f)$

- For any two partitions P_1 and P_2,

$$L(P_1,f) \le U(P_2,f)$$

i.e. no upper sum can be less than any lower sum.

- For partition *P* of [*a, b*] and points $t_1, t_2, ..., t_n$ such that $x_{i-1} \le t_i \le x_i, \, (i = 1,2,...,n),$ the sum

$$S(P,F) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

is said to be Riemann sum of *f* over [*a, b*] relative to *P*.

- The second definition of integrability : A function *f* is said to be integrable on [*a, b*] if $\lim_{\mu(P)\to 0} S(P,f)$ exists as $\mu(P) \to 0$ and

$$\lim_{\mu(P)\to 0} S(P,f) = \int_a^b f dx$$

- A necessary and sufficient condition for the integrability of a bounded function *f* is that to every $\in > 0,$ there corresponds $\delta > 0$ such that for every partition *P* of [*a, b*] with norm $\mu(P) < \delta,$

  $U(P,f) - L(P,f) < \in$

- A bounded function *f* is integrable on [*a, b*] if and only if for every $\in > 0$, there exists a partition *P* of [*a, b*] such that

  $U(P,f) - L(P,f) < \in$

- Every continuous function on [*a, b*] is integrable on [*a, b*].
- If function *f* is bounded and integrable on [*a, b*], then the function *F* defined as

  $F(x) = \int_a^x f(t)dt,\ a \le x \le b$

  is continuous on [*a, b*], and furthermore, if *f* is continuous at a point $c \in [a,b],$ then *F* is derivable at *c* and

  $F'(c) = f(c)$

- /If a function *f* is bounded and integrable on [*a, b*] and there exists a function *F* such that $F' = f$ on [*a, b*], then,

  $\int_a^b f dx = F(b) - f(a)$



11.9 ANSWERS TO CHECK YOUR PROGRESS

Check your Progress 1

Ans to Q No 1: Try Yourself.

Check your Progress 2

Ans to Q No 1: The function *f* is bounded and continuous on [– 1, 1] and

$$f(x) = \begin{cases} -x, & xleq 0 \\ x & x \ge 0 \end{cases}$$

Let the partition $P = \{-1 = x_0, x_1, ..., x_n = 0 = y_0, y_1, ..., y_n = 1\}$ devides [– 1, 1] into 2*n* equal sub-intervals each of length 1/*n*, so that

$\mu(P) = 1/n \to 0$ as $n \to \infty$

$$x_i = -1 + i/n,\ i = 1,2,...,n$$

$$y_i = 1/i,\ i = 1,2,...,n$$

$$\Delta x_i = 1/n = \Delta y_i$$

$$\sum_{i=1}^{n} \Delta x_i = 1 = \sum_{i=1}^{n} \Delta y_i$$

Let $t_i \in \Delta x_i$ and $t_i' \in \Delta y_i$ and let $t_i = x_i$ and $t_i' = y_i$, for $i = 1,2,...,n$

$$S(P,f) = \sum_{i=1}^{n} f(t_i)\Delta x_i + \sum_{i=1}^{n} f(t_i')\Delta y_i$$

$$= \sum_{i}^{i=1} (-x_i)\Delta x_i + \sum_{i}^{i=1} y_i \Delta y_i$$

$$= \sum_{i} \left(1 - \frac{i}{n}\right) \Delta x_i + \sum_{i=1}^{n} \frac{i}{n} \Delta y_i$$

$$= \sum_{i=1}^{n} \Delta x_i - \frac{1}{n^2} \sum_{i=1}^{n} i + \frac{1}{n^2} \sum_{i=1}^{n} i$$

Thus $\lim_{\mu(P)\to 0} S(P,f) = 1$

and since theh limit exists, the function is integrable and

$$\int_{-1}^{1} |x|\, dx = \lim S(P,f) = 1$$

Ans to Q No 2: Try Yourself.



11.10 FURTHER READING

1) Walter, R. Principles of Mathematical Analysis. 3rd edition, McGraw-Hill, 2017.
2) Terence T. Analysis II. Hindustan Book Agency, 2009.
3) Malik, S.C. and Arora, S. Mathematical Analysis. 2nd edition reprint. New Age International Publishers, 2005.
4) Apostol, T.M. Mathematical Analysis. 2nd edition. Wesley Publishing Co. 2002.
5) Somasundram, D. and Chaudhary, B. A First Course in Mathematical Analysis. Narosa Publishing House, 1996.

6) Royden, H. L. Real Analysis, Macmillan Pub. Co., Inc. 4th edition, New York, 1993.

7) Bartle, R.G., Sherbert, D. R. (2011). Introduction to real analysis. Hoboken, NJ: Wiley.

8) Narayan, S., Raisinghania, M.D. Elements of Real Analysis. S. Chand & Company Pvt. Ltd. 2015.



11.11 MODEL QUESTIONS

Q 1: Prove that the function *f* defined as

$$f(x)=\frac{1}{2^n},\ \text{when}\ \frac{1}{2^n+1}<x\le\frac{1}{2^n},\ (n=0,1,2,...)$$

and *f* (0) = 0 is integrable on [0, 1] although it has an infinite number of points of discontinuity.

Q 2: Show that the function |*x*|, where |*x*| denote the greatest integer function is integrable on [0, 3] and

$$\int_0^3 |x|\,dx=3$$

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